

# The Theorem of Ehlich and Zeller to Develop Polytopic Models for Set-Based Observers

Andreas Rauh and Marit Lahme

Carl von Ossietzky Universität Oldenburg  
Distributed Control in Interconnected Systems  
D-26111 Oldenburg, Germany

{Andreas.Rauh, Marit.Lahme}@uni-oldenburg.de

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## Interval Observer Design

Interval observers provide a means to perform state estimation in a bounded error framework. For that purpose, the design is typically performed in such a way that lower and upper bounding trajectories are determined so that the sets of reachable states are enclosed at each point in time on the basis of model knowledge and measurements with bounded uncertainty [2, 6].

Originally, this design was developed for linear time-invariant system models in the form

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{w}(t) \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{v}(t) \quad , \quad (2)$$

where  $\mathbf{x}(t) \in \mathbb{R}^n$  is the state vector,  $\mathbf{u}(t) \in \mathbb{R}^m$  the control vector, and  $\mathbf{y}(t) \in \mathbb{R}^q$  the vector of measurements. In addition,  $\mathbf{w}(t)$  and  $\mathbf{v}(t)$  are disturbance vectors that are assumed to be bounded according to  $\mathbf{w}(t) \in [\underline{\mathbf{w}}(t) ; \overline{\mathbf{w}}(t)]$  and  $\mathbf{v}(t) \in [\underline{\mathbf{v}}(t) ; \overline{\mathbf{v}}(t)]$ , respectively.

For the system model (1) with the actual sensor reading  $\mathbf{y}_m(t)$ , an interval observer is then given in the form

$$\dot{\underline{\hat{\mathbf{x}}}}(t) = \mathbf{A} \cdot \underline{\hat{\mathbf{x}}}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \underline{\mathbf{w}}(t) + \mathbf{H} \cdot (\mathbf{y}_m(t) + \underline{\mathbf{v}}(t) - \mathbf{C} \cdot \underline{\hat{\mathbf{x}}}(t)) \quad (3)$$

$$\dot{\overline{\hat{\mathbf{x}}}}(t) = \mathbf{A} \cdot \overline{\hat{\mathbf{x}}}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \overline{\mathbf{w}}(t) + \mathbf{H} \cdot (\mathbf{y}_m(t) + \overline{\mathbf{v}}(t) - \mathbf{C} \cdot \overline{\hat{\mathbf{x}}}(t)) \quad (4)$$

if the following assumptions are satisfied:

- the lower state bound  $\underline{\hat{\mathbf{x}}}(t)$  is element-wise non-negative;
- all off-diagonal elements of the matrix  $\mathbf{A} - \mathbf{H}\mathbf{C}$  are non-negative, i.e., it is a Metzler matrix;
- $\mathbf{A} - \mathbf{H}\mathbf{C}$  is Hurwitz, i.e., the bounding trajectories are stable;
- the term  $\mathbf{B} \cdot \mathbf{u}(t)$  is element-wise non-negative; and
- the true initial state of the system satisfies the relation  $\mathbf{x}(0) \in [\underline{\hat{\mathbf{x}}}(0) ; \overline{\hat{\mathbf{x}}}(0)]$ .

Generalizations were developed in recent years for systems that have bounded uncertainty in the matrices  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  and for systems with negative bounds  $\underline{\hat{\mathbf{x}}}(t)$  in (some) of the state vector components. Moreover, the restriction of  $\mathbf{A} - \mathbf{H}\mathbf{C}$  being a Metzler matrix was relaxed by either coordinate transformations [4] or by extending the observer differential equations (3) and (4) by two further gain matrices in addition to the gain  $\mathbf{H}$ , cf. [7].

The uncertainty in  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  can either be bounded and temporally constant or varying within predefined bounds. In both cases, design conditions for the observer gain can be derived in the form of linear matrix inequalities (LMIs), where the uncertainty in the matrices is treated in the form of a polytopic uncertainty model, leading to an evaluation of the LMIs at all possible combinations of the extremal values of these matrices.

## Generalization to Nonlinear System Models

For the case of nonlinear system models, a polytopic uncertainty model needs to be determined as mentioned before if an observer is designed so that the bounding trajectories are determined on the basis of a quasi-linear system formulation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t)) \cdot \mathbf{u}(t) + \mathbf{w}(t) \quad (5)$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{x}(t)) \cdot \mathbf{x}(t) + \mathbf{v}(t) \quad (6)$$

However, a naive determination of a polytopic enclosure of all of the state-dependent matrices in the quasi-linear model (5) easily leads to enclosing many combinations of values that are not at all attainable during the evolution of the system.

Therefore, it is necessary that the polytopic enclosure of all system matrices is determined with small pessimism. For that purpose, the theorem of Ehlich and Zeller can be employed if the nonlinearities within these matrices are given in polynomial form [3]. An application of this theorem was recently published in [1], where a combined optimization of the gains of observers and controllers was performed so that stability is ensured in cases in which the classical separation principle of control and observer design no longer holds and the sensitivity against stochastic noise is reduced simultaneously.

As many real-life systems have nonlinearities within the system matrices that are not pure polynomials, we propose the following procedure in this contribution:

- determine a set-based enclosure for each matrix element in the form of a polynomial approximation with freely choosable degree with additive interval bounds for these polynomial approximations so that the true nonlinearities are included for all possible operating states;
- employ the theorem of Ehlich and Zeller to determine a polytopic approximation capable of capturing correlations between each of the matrix elements and between the matrices;

- solve the design LMIs for the interval observer synthesis.

The resulting procedure is then demonstrated for the state estimation of a simplified model of a lithium-ion battery [5].

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