A hybrid algorithm to solve Chebyshev approximation problem with interval certification

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Introduction

Chebyshev approximation problem [3] of the form,

$$\min_{a\in\mathbb{R}^n}\max_{x\in X}|e(a,x)| \tag{1}$$

where $e(a,x) = P_a(x) - f(x)$ is the error function, $P_a : X \to \mathbb{R}^d$ is the polynomial of coefficients $a \in \mathbb{R}^n$ that approximate a function $f : X \to \mathbb{R}^d$ for $d, n \in \mathbb{N}^*$.

This study aims to design efficient algorithms to solve multivariate Chebyshev approximation. A Chebyshev approximation problem is univariate when X is a interval in $\mathbb{R}$ and multivariate when X is a box in $\mathbb{R}^d$. For example the function $f(x) = (1 + x_1)^{x_2}$ is shown in Figure 1 on the left next to the error function associated to its approximation $P_a(x) = 2.25 - 2.71x_1 + 1.06x_1^2 - 1.44x_2 + 0.36x_2^2 + 2.49x_1x_2$. This algorithmic field remains not much explored because of the difficulties of the problem as well as a lack of applications. While a lot of mathematical work was conducted, few algorithmic work were made. Furthermore, much of the field still relies on the Remez algorithm [1] made in 1934. Other algorithmic studies were later made in the 1990s by Reemtsen with his work on the Remez algorithm and methods of discretization [2] to solve semi-infinite problems applied to Chebyshev approximation problems [3].

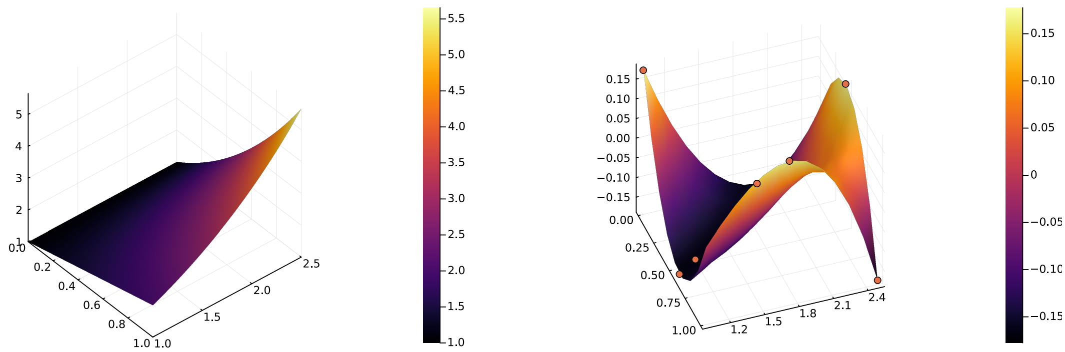

Figure 1: Illustration of the example : left) The plot of f; right) the error function associated to its approximation (with the extrema)

Algorithms

The two-phase method is investigated, and a hybrid algorithm is proposed. An implementation for both of them was made for solving multivariate Chebyshev approximation problems. The two-phase algorithm operates by first discretizing the Chebyshev problem 1 which yield a initial solution then in a second time a Newton algorithm is applied to the initial solution to adjust it to optimality. This method is capable of identifying optimal approximation polynomials while being non-iterative; however, its success heavily depends on the quality of the discretization. Poor discretizations often lead to suboptimal results. Preliminary experiments using Reemtsen's functions [2] demonstrate that our approach achieves comparable levels of optimality certification to existing methods.

The hybrid algorithm combines elements of the second Remez algorithm and of the two-phase algorithm. This algorithm works iteratively by finding an approximation with the same discretization as in the two-phase method, then refining it with local optimization, and finally certifying it (see next Section), which reduces the number of iterations until convergence and the resolution time for the Reemtsen's functions compared to the Remez algorithm.

Certification

To certify the solution of the hybrid algorithm, we compute an enclosure $[\underline{t}, \overline{t}]$ of the maximal approximation error.

The lower bound

The formulation of Chebyshev approximation problem 1 can be rewritten as a semi-infinite linear problem (SILP), from which the infinite set X is discretized, yielding $\hat{X}$ containing a finite number of samples which are used to formulate the following relaxed linear problem:

$$\begin{aligned} \min_{t \in \mathbb{R}, a \in \mathbb{R}^n} \quad & t \\ \text{s.t.} \quad & P_a(x) - f(x) \leq t, \forall x \in \hat{X} \\ & -(P_a(x) - f(x)) \leq t, \forall x \in \hat{X} \end{aligned} \tag{2}$$

The minimizer $\underline{t}$ of problem 2 is a lower bound for problem 1.

The upper bound

The upper bound is found with a non-linear interval branch-and-prune that solves problem 3 where the parameter $a \in \mathbb{R}^n$ are fixed.

$$\begin{aligned} \max_{t \in \mathbb{R}, x \in X} \quad & t \\ \text{s.t.} \quad & t \leq |e(a, x)| \end{aligned} \tag{3}$$

The Ibex solver [4] is used to solve this problem. The maximizer $\overline{t}$ of problem 3 is a upper bound for problem 1.

References

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